A Structural Model of Dense Network Formation

Asymptotic Results

Mele (2017)

Reading Group on Stochastic Models

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Previous Sections...

- So far, the likelihood and power function has been described as:

$$\pi_n(g) = \frac{\exp\left[\sum_{i=1}^n \sum_{j=1}^n g_{ij} u_{ij} + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n g_{ij} g_{ji} m_{ij} + \sum_{i=1}^n \sum_{j=1}^n \sum_{k \neq i,j}^n g_{ij} g_{jk} v_{ij}\right]}{c(\theta, \mathcal{G}_n)}$$

- In order to apply asymptotic analysis, this can be rewritten:

$$\pi_n(g) = \frac{\exp\left\{ n^2 \left[\alpha \frac{\sum_{i=1}^n \sum_{j=1}^n g_{ij}}{n^2} + \frac{\beta}{2} \frac{\sum_{i=1}^n \sum_{j=1}^n g_{ij} g_{ji}}{n^2} + \gamma \sum_{i=1}^n \sum_{j=1}^n \frac{\sum_{k \neq i,j}^n g_{ij} g_{jk}}{n^3} \right] \right\}}{c(\alpha, \beta, \gamma, \mathcal{G}_n)}$$

  - Main assumption:
    - Completely homogeneous agents.
    - Notice that $\gamma^0 = \frac{\gamma}{n}$.

Notions on Graph Limits

From Lovasz (2012):

- Given a sequence $\{G_n\}_{n\geq 0}$, we are interested on the homomorphism density as $n \to \infty$:

$$t(H, G) = \frac{|hom(H, G)|}{|V(G)|^{|V(H)|}} \quad (1)$$

- If $t(H, G_n)$ converges, it converges to the limit object:

$$t(H, h) = \int_{[0,1]^k} \prod_{(i,j)\in E(H)} h(x_i, x_j) dx_1 ... dx_k, \quad (2)$$

  where $h \in \mathcal{W}$ (all measurable functions $h : [0, 1]^2 \to [0, 1]$).
- Relation with exchangeability?
- Defining the appropriate distance ($\delta_{\square}(f, g)$) and a quotient space ($\widetilde{\mathcal{W}}$), compactness of the set and continuity of $t(\cdot, \cdot)$ are achieved.

Notions on Graph Limits

- The stationary distribution can be expressed as:

$$\pi_n(g, \alpha, \beta) = \exp\{n^2[\mathcal{T}(\widetilde{G}) - \psi_n]\}, \quad (3)$$

- The term ψ_n is the normalizing constant:

$$\psi_n = \frac{1}{n^2} \log \sum_{G \in \mathcal{G}_n} \exp\left\{ n^2[\mathcal{T}(\widetilde{G})] \right\}. \quad (4)$$

Notions on Large Deviation for Random Graphs

From Chatterjee and Varadhan (2011):

- Based on a rate function:

$$\mathcal{I}(u) = u \log u + (1-u)\log(1-u)$$

- Extended to $\widetilde{\mathcal{W}}$:

$$\mathcal{I}(\widetilde{h}) = \int_0^1 \int_0^1 \mathcal{I}(h(x,y))dxdy \tag{5}$$

Theorem 10 (Asymptotic Log-Constant for Directed Graphs)

If $\mathcal{T} : \widetilde{\mathcal{W}} \to \mathbb{R}$ is a bounded continuous function and ψ_n and $\mathcal{I}$ are defined as before:

$$\psi \equiv \lim_{n\to\infty} \psi_n = \sup_{\widetilde{h}\in\widetilde{W}} \left\{ \mathcal{T}(\widetilde{h}) - \mathcal{I}(\widetilde{h}) \right\} \tag{6}$$

Application to Mele (2017)

Theorem 2 (Non-negative Link Externalities)

Model (3) with non-negative link externalities $\beta \geq 0$ exhibits the following behavior:

1. asymptotic normalizing constant ψ solves:

$$\psi \equiv \lim_{n\to\infty} \psi_n = \max_{\mu\in[0,1]} \left\{ \alpha\mu + \beta\mu^2 - \mu \log(\mu) - (1-\mu)\log(1-\mu) \right\} \quad (7)$$

2. networks generated by the model are indistinguishable from directed Erdos-Renyi graph with linking probability μ^*, defined as follow:
   (a) if maximization of (7) has a unique solution, then μ^* satisfies $2\beta\mu(1-\mu) < 1$ for almost all $\alpha \in \mathbb{R}$ and $\beta \geq 0$ and solves:

$$\mu = \frac{\exp\{\alpha + 2\beta\mu\}}{1 + \exp\{\alpha + 2\beta\mu\}} \quad (8)$$

   (b) if the maximization of (7) has two solutions, then μ^* picked randomly from some probability distribution over μ_1^* and μ_2^*, s.t. $\mu_1^* < 0.5 < \mu_2^*$, and both solve (7) and satisfy $2\beta\mu(1-\mu) < 1$.

Application to Mele (2017)

Theorem 3 (Negative Link Externalities)

If $\beta < 0$ and sufficiently large in magnitude, model (3) is asymptotically different from a directed Erdos-Renyi model.

- The model would be different from the directed Erdos-Renyi model with at least one negative externality (i.e. $\beta < 0$ or $\gamma < 0$).
- V-shape region has two global maximum and it varies with the level of utility.